Year 12 Solutions: Week 1.

(1.) $x = 24 \times 150 = (2^3 \cdot 3) \times (10 \times 15) = 2^3 \cdot 3\,(2 \cdot 5^2 \cdot 3) = 2^4 \cdot 3^2 \cdot 5^2$. $\therefore \sqrt{x} = 2^2 \cdot 3 \cdot 5 = 60.$ //

$y = 147 \times 48 = (3\,(49)) \cdot 2^2\,(2^2 \cdot 3) = 2^4 \cdot 3^2 \cdot 7^2$. $\therefore \sqrt{y} = 2^2 \cdot 3 \cdot 7 = 84.$ //

$z = a^2 \cdot b^2 \cdot c^6$. $\therefore \sqrt{z} = a \cdot b \cdot c^3.$ //

(2.) $n^3 - n = n(n^2-1) = (n-1) \cdot n \cdot (n+1)$. $\forall\, n$ one of $(n-1)$, n or $(n+1)$ is divisible by 3. //

(3.) N.B. $0 \leq \sin^2(10x+11) \leq 1 \quad \forall\, x \in \mathbb{R}$.

$\therefore\ 36 \leq f(x) \leq 49. \quad \Rightarrow \quad$ (c.) 49. //

(4.) $\begin{array}{r} 168 \\ \times 3 \\ \hline 504. \end{array}$ (d.) x^{504}. //

(5.) a.) $x_1 = 1,\ y_1 = 0$ OR $x_1 = 3,\ y_1 = 2.$ //

(b.) $x_{n+1} = 3x_n + 4y_n, \quad y_{n+1} = ax_n + by_n$ for $n \geq 1$.

$$(x_{n+1})^2 - 2(y_{n+1})^2 = 9x_n^2 + 16y_n^2 + 24x_ny_n - 2(a^2x_n^2 + b^2y_n^2 + 2ab\,x_ny_n)$$

$$= (9-2a^2)x_n^2 + (16-2b^2)y_n^2 + (24-4ab)x_ny_n$$

$$\equiv x_n^2 - 2y_n^2$$

$\Rightarrow\ 9 - 2a^2 = 1. \quad \Rightarrow\ a^2 = 2 \quad \Rightarrow\ a = 2$ // as $a \geq 0$.

$16 - 2b^2 = -2. \quad \Rightarrow\ b^2 = 9 \quad \Rightarrow\ b = 3$ // as $b \geq 0$.

$24 = 4ab. \quad \Rightarrow\ a \cdot b = 6.$ (✓).

i.e. we have: $x_{n+1} = 3x_n + 4y_n, \quad y_{n+1} = 2x_n + 3y_n.$

(c.) We seek $X^2 - 2Y^2 = 1$ and we know that $x_n^2 - 2y_n^2 = (x_{n+1})^2 - 2(y_{n+1})^2$

where $x_{n+1} = 3x_n + 4y_n$, $y_{n+1} = 2x_n + 3y_n$.

From part (a.) we have $x_0 = 1$, $y_0 = 0$ then:

	$n=0.$	$1.$	$2.$	$3.$
x_n	1.	3.	17.	99.
y_n	0.	2.	12.	70.

$\Rightarrow$ $X = 99$ //, $Y = 70$ // satisfies $X^2 - 2Y^2 = 1$ for $X > Y > 50$.

(d.) $\lim_{n\to\infty} (x_n^2 - 2y_n^2) = 1.$

$\therefore \lim_{n\to\infty} \left(\frac{x_n^2}{y_n^2} - 2 \right) = \lim_{n\to\infty} \left(\frac{1}{y_{n+1}^2} \right) = 0 \Rightarrow \lim_{n\to\infty} \left(\frac{x_n}{y_n} \right) = \sqrt{2}.$ //

(5.) a.) Let $f(x) = x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y.$

then $f(3) = \cancel{27} - \cancel{45} + \cancel{18y} + \cancel{3y^2} - \cancel{24y} - \cancel{3y^2} + \cancel{18} + \cancel{6y}. = 0.$

$\therefore f(x) = (x-3)(x^2 - 2x + y^2 + 2xy - 2y)$

$\equiv (x-3)(x + ay + b)(x + cy + d)$

$= (x-3)(x^2 + cxy + dx + ayx + acy^2 + ady + bx + bcy + bd).$

$\Rightarrow$ $d + b = -2$, $ac = 1$, $c + a = 2$, $ad + bc = -2$, $bd = 0.$

$\Rightarrow a(2-a) = 1 \Rightarrow a = 1$ as $a \in \mathbb{Z}$

$\Rightarrow c = 1.$

$\Rightarrow b(-2-b) = 0 \Rightarrow b = 0 \Rightarrow d = -2$ or $b = -2 \Rightarrow d = 0.$

$\therefore f(x) = (x-3)(x+y)(x+y-2).$

(b.) Looking at $2x^2 + x^2y$ suggest a factor of $(y+2)$:

$(y+2)(6y^2 - 13y + 5 + x^2 - 5xy + 6x)$

$= (y+2)(ay + bx + 1)(cy + dx + 5)$

$\Rightarrow \underset{y^2}{ac = 6}, \quad \underset{xy}{ad + cb = -5}, \quad \underset{y}{5a + c = -13}, \quad \underset{x^2}{bd = 1}, \quad \underset{x}{5b + d = 6}.$

$\Rightarrow b(6-5b) = 1 \quad \Rightarrow b = 1$ if $b \in \mathbb{Z}$.

$\Rightarrow d = 1.$

$\Rightarrow \left.\begin{array}{l} a + c = -5 \\ 5a + c = -13 \end{array}\right\} \quad 4a = -8 \quad \Rightarrow a = -2.$

$\Rightarrow c = -3.$

i.e. we have $(y+2)(-2y + x + 1)(-3y + x + 5).$