This file contains hints for questions set in Year 13 tKF sessions, to give you a nudge if you feel really stuck Use the hints a step at a time, if the first step does not help, move on to the next. Use as few lines as possible before completing the question yourself.

1. To calculate the square root of 24×150 break 24 and 150 down into factors until all factors are either prefect squares or occur twice.

   Further hint: $24 \times 150 = 4 \times 6 \times 6 \times 25$.

   Alternatively, you can factorise 24×150 into prime factors as $2 \times 2 \times 2 \times 3 \quad \times \quad 2 \times 3 \times 5 \times 5 = 2^4 \times 3^2 \times 5^2$.

2. This can be proved by induction.

   Alternatively, note that any integer n can be expressed as either $3m$ or $3m+1$ or $3m+2$ for some integer m.

   Taking, for example, $n = 3m+1$, show that $(3m+1)^3 - (3m+1)$ is a multiple of 3. (Repeat for the other two cases.)

3. [2008 Oxford Admission Test question 1E] (Multiple Choice)

   Try this a bit at a time, until you can see how the highest power will emerge:

   What is the highest power of x in

   $$f(x) = \left(2x^6 + 7\right)^3 ?$$

   What is the highest power of x in

   $$f(x) = \left(2x^6 + 7\right)^3 + \left(3x^8 - 12\right)^4 ?$$

   What is the highest power of x in

   $$f(x) = \left[\left(2x^6 + 7\right)^3 + \left(3x^8 - 12\right)^4\right]^5 ?$$

   etc etc

4. [STEP 1 2004 qu 1]

(a) You can work this out in two stages, first calculating $(3+2\sqrt{5})^2$ and then multiplying by $(3+2\sqrt{5})$. You can also work it out using the binomial theorem, but it is probably worth learning the first few rows of Pascal's triangle, including the row

$$1 \quad 3 \quad 3 \quad 1\,,$$

which immediately gives you $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

(b) Use the condition $\sqrt[3]{99-70\sqrt{2}} = c - d\sqrt{2}$ to write down two simultaneous equations satisfied by c and d. Rather than trying to solve these equations analytically, read the question carefully [IMPORTANT PRINCIPLE: make sure you use all the information given in a question] and notice that c and d are positive integers. Look at the equations until you see that there are very limited possibilities for c and d, and then use trial and error. (Make sure both equations are satisfied.)

(c) This looks daunting as it has x^6 in it, but the only other power is x^3 so you can set $y = x^3$ giving the quadratic equation $y^2 - 198y^3 + 1 = 0$. Solve this, to get $y = 99 \pm 70\sqrt{2}$. Then use the previous part to solve for x.

5. [2008 Oxford Admission Test question 2]

(a) To find a pair of positive integers x_1 and y_1, that solve the equation $(x_1)^2 - 2(y_1)^2 = 1$ simply try out some small integers.

(b)

$$\text{Substitute} \quad x_{n+1} = 3x_n + 4y_n \quad \text{and} \quad y_{n+1} = ax_n + by_n$$

$$\text{into} \quad (x_n)^2 - 2(y_n)^2 = (x_{n+1})^2 - 2(y_{n+1})^2\,.$$

and match coefficients.

(c) Using your values of x_1, y_1 from part (a), and the recurrence relations

$$x_{n+1} = 3x_n + 4y_n \quad \text{and} \quad y_{n+1} = ax_n + by_n$$

(with values of a and b from part (b)) evaluate x_2 and y_2 and explain why they satisfy

$$(x_2)^2 - 2(y_2)^2 = 1\,.$$

Repeat this to calculate x_3, y_3 and so on until both exceed 50.

(d) Rearrange the equation $(x_n)^2 - 2(y_n)^2 = 1$. (Why does this equation hold?)

6. [2004 STEP I question 3]

(a) To show that $x-3$ is a factor of

$$g(x) = x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y\,. \qquad (*)$$

show that $g(3) = 0$ and apply the factor theorem.
Next note that

$$g(x) = x^3 - 5x^2 + 6x + y(2x^2 - 8x + 6) + y^2(x-3)$$

and factorise $x^3 - 5x^2 + 6x$ and $2x^2 - 8x + 6$.

(b) To factorise $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ first find a factor of the form $y + c$.

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